Analytical Analysis and Numerical Solution of Two Flavours Skyrmion*

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Abstract

Two flavours Skyrmion will be analyzed analytically, in case of static and rotational Skyrme equations. Numerical solution of a nonlinear scalar field equation, i.e. the Skyrme equation, will be worked with finite difference method. This article is a more comprehensive version of *SU(2) Skyrme Model for Hadron* which have been published at Journal of Theoretical and Computational Studies, Volume **3** (2004) 0407.

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1 Introduction

Nonlinear physics is a general phenomenon in physics. It includes very long range from e.g. particle, nuclear, condensed matter, fluids, plasmas, biophysics (for example: nonlinear diffusion-reaction, DNA), to cosmology.

Soliton is defined as classical solution of nonlinear wave equation which has properties: finite total energy, localized, nondispersive, stable with its profile of energy density distribution is like pulse centered in finite space [1].

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Skyrmion is a *topological soliton* in three dimensions. Following, we will discuss soliton as *baryon model*, especially *Skyrme model* for hadron which is formulated in *extended nonlinear Sigma model* [1], [2].

2 Topological Solitons

Suppose that $F_\phi = 0$ is a differential equation or a system of differential equation involving a field or a set of fields denoted by ϕ which are functions of D space coordinates $\vec{x}$ and a time coordinate t. We shall assume that $F_\phi = 0$ is the equation of motion which results from some Lagrangian filed theory. We also assume that the system has a corresponding "energy" ε_ϕ and "energy density" $E_\phi(\vec{x}, t)$ [3]

$$\varepsilon_\phi = \int d^D x \; E_\phi(\vec{x}, t) \tag{1}$$

where for all allowed field configurations ϕ, E_ϕ is greater than or equal to zero. If E_ϕ is zero for all $\vec{x}$, we call ϕ *a ground state* or *a vacuum solution*, and denote it by ϕ_{vac}. The vacuum solution need not be unique.

Now suppose that $\phi = \phi_{cl}$ is *non-vacuum solution* to $F_\phi = 0$. Following Coleman, we shall call $\phi_{cl}(\vec{x}, t)$ a *soliton solution* if the following properties are satisfied [3]:

(1) $\varepsilon_{\phi=\phi_{cl}}$ is *finite*.

(2) $E_{\phi=\phi_{cl}}(\vec{x}, t)$ is *nonsingular (finite)* for all value of $\vec{x}$ and t, and *localized* for all times t. A solution is localized at any time t if there is a bounded region of space defined by $E_{\phi=\phi_{cl}}(\vec{x}, t) \geq \delta$, δ being any arbitrary number, fulfilling

$$0 < \delta < \max_{\vec{x}} \; E_{\phi=\phi_{cl}}(\vec{x}, t). \tag{2}$$

We say that the solution is localized for all times t if the bounded region can be chosen independent of t.

(3) $\phi_{cl}(\vec{x}, t)$ is *nonsingular*.

(4) $\phi_{cl}(\vec{x}, t)$ is *nondissipative*.
Concerning (4), a solution $\phi_{cl}(\vec{x}, t)$ is considered dissipative if

$$\lim_{t \to \infty} \max_{\vec{x}} \; E_{\phi=\phi_{cl}}(\vec{x}, t) = 0. \tag{3}$$

In our discussions, we will be concerned primarily with "static" solutions for which we denote $\phi_{cl}(\vec{x}, t)$ by $\Phi_{cl}(\vec{x}, t)$.

If we exclude the vacuum, then static, nonsingular, localized solutions are automatically nondissipative. Thus for static solitons, we only require (1), (2) and (3).

We will, however, be interested in static solutions which satisfy yet another requirements; namely that they be

2

(5) *classically stable.* We call a static solution "classically stable" if under all possible infinitesimal fluctuations $\delta\phi$ of the fields ϕ about the static solution,

$$\varepsilon_{\phi_{cl}+\delta\phi} \geq \varepsilon_{\phi_{cl}}. \tag{4}$$

Variational modes $\delta_0\phi$ which leave the energy unchanged are said to be "zero" frequency modes" or "zero modes". Thus, for zero modes, $\varepsilon_{\phi_{cl}+\delta_0\phi} = \varepsilon_{\phi_{cl}}$. *Zero modes are generally associated with various symmetries of the soliton.* Excluding the zero modes, a classically stable solution is a local minimum of the energy.

Although the question of *stability* is ultimately a dynamical one, *topology* often plays an important role. We shall now elaborate on this point. Let Q be *the set* of all finite energy and nonsingular field configurations ϕ at some fixed time t. It is thus *the configuration space* of the system. *A subset Q_1 of Q is said to be (path-)connected* if any field ϕ_1 of Q_1 can be *continuously deformed* to any other field ϕ_1' in Q_1.

Two subsets Q_1 and Q_2 of Q are said to be *disconnected* if any field ϕ_1 of Q_1 cannot be continuously deformed to any field ϕ_2 in Q_2. We shall consider the case where Q has $N > 1$ disconnected components Q_n, each Q_n being connected. Q is the union of all these disconnected components,

$$Q = \cup_{n=1}^{N} Q_n. \tag{5}$$

Let $\phi(\vec{x})$ and $\phi'(\vec{x})$ be two *fields* in Q. $\phi(\vec{x})$ and $\phi'(\vec{x})$ are *homotopic* to each other, and write $\phi \sim \phi'$, if there exists a sequence of fields $\phi^{(\tau)}(\vec{x})$, $0 \leq \tau \leq 1$, which is continuous in τ and $\vec{x}$, with $\phi^{(0)}(\vec{x}) = \phi(\vec{x})$ and $\phi^{(1)}(\vec{x}) = \phi'(\vec{x})$. With this definition, all fields within one component Q_n are "homotopic" to each other, while a field $\phi^{(n)}$ belonging to Q_n is not homotopic to a field $\phi^{(n')}$ belonging to $Q_{n'}$ when $n' \neq n$ [3].

We can treat all our examples assuming that n is countable. We make this assumption: *What is the physical significance of this classification?* Consider the initial conditions $(\phi^{(n)}, d\phi^{(n)}/dt)$ at time $t = 0$ for the equations of motion. (These equations for simplicity are assumed to be second order in time). Assume that $\phi^{(n)} \in Q_n$. After a lapse of time T, suppose that $\phi^{(n)}$ becomes $\phi'^{(n)}$ and $d\phi^{(n)}/dt$ becomes $d\phi'^{(n)}/dt$. Since time evolution is assumed to be a continuous operation, it follows that $\phi'^{(n)}$ is homotopic to $\phi^{(n)}$. Hence, $\phi'^{(n)} \in Q_n$. In other words, the value of n associated with the field $\phi^{(n)}$ is a constant of the motion. Hence, if we define Q_0 to be the sector which contains the vacuum solution ϕ_{vac}, then the configurations $\phi^{(n)} \in Q_n$, $n \neq 0$, cannot be time evolved to ϕ_{vac} or in fact, to any other $\phi^{(m)} \in Q_m$ with $m \neq n$. For such reasons, the sectors $Q_{n\neq0}$ of field configurations are said to be *topological stable*. Soliton configurations which fulfill the five properties listed previously and which are in topologically stable sectors are called *topological solitons* [3].

3   Two Flavours Skyrmion as Hadron

In particle physics, *flavour* is a quantum number of elementary particles. In quantum chromodynamics (QCD), flavour is a global symmetry. In the electroweak theory, on the other hand, this symmetry is broken, and flavour changing processes exist [4].

If there are two or more particles which have identical interactions, then they may be interchanged without affecting the physics. Any (complex) linear combination of these two

particles give the same physics, as long as they are orthogonal or perpendicular to each other. In other words, the theory possesses symmetry transformations such as $M \begin{pmatrix} u \\ d \end{pmatrix}$, where u and d are the two fields, and M is any 2×2 unitary matrix with a unit determinant. Such matrices form a Lie group called $SU(2)$, *special unitary group.* This is an example of flavour symmetry. The term "flavour" was first coined for use in the quark model of hadrons in 1968 [4].

In order to understand the charge radius of nucleon which have size roughly 1 Fermi, Tony Hilton Royle Skyrme in 1962 proposed idea that strongly interacting particles (hadrons) were locally concentrated static solution of extended nonlinear Sigma (Chiral) model. In (3+1)-dimensions of space-time, we will observe Skyrme model which describes hadrons as solitons (Skyrmions) from nonlinear Sigma (chiral) model field theory in internal symmetrical group of SU(2): two flavours Skyrmion.

Skyrme's idea is unifying bosons and fermions in a common framework which provide a fundamental fields model consisted of only pion. The nucleon was obtained, as a certain classical configuration of the pion fields. Skyrmion is a topological soliton object, i.e. solution to the classical field equation with localized energy density. Various atoms should correspond to vortices of different connectivities in some underlying liquid.

Following, some ideas which are proposed by Skyrme in relations with baryon model [1], [8], [12]:

(1) Meson field can take its value in S_3 manifold. As the result of this assumption, Skyrme discovers conserved quatity i.e. *topological charge* or *winding number* and he interpretates it as *baryon number.*

(2) Solution of Skyrme field equation in spherical coordinate takes form:

$$F_a(r, \theta, \phi) = g(r)n_a(\theta, \phi) \tag{6}$$

where $g(r)$ is profile function which has spherical symmetric character and n_a ($a = 1, 2, 3$) is component of unit vector, $\widehat{\boldsymbol{n}}$. $F_a(r, \theta, \phi)$ is called *Skyrme ansatz* or *hedgehog.*

(3) Skyrme ansatz describes *stable extended particle* with unit topological charge.

(4) Skyrme ansatz with unit topological charge *can be quantized* as fermion, so it is possible to identify state of isotopic spin, I, and total spin, J, with nucleon doublet in case of $I = J = \frac{1}{2}$, and with Δ resonance in case of $I = J = \frac{3}{2}$.

4 SU(2) Skyrme Model

SU(2) Skyrme model is very simple model, because it only consists of two flavours. This model is described by $U = U(x^\mu)$ function which has SU(2) group valued of (3+1)D space-time coordinates [1, 12, 14].

The dynamics is determined by *action*

$$S = \int d^4x \, \mathcal{L}, \tag{7}$$

where

$$\mathcal{L} = \text{Tr}\left[-\frac{F^2}{16}L_\mu L^\mu + \frac{1}{32a^2}\left[L_\mu, L_\nu\right]\left[L^\mu, L^\nu\right] + \frac{F^2}{16}M_\pi^2(U^{-1} + U - 2I)\right] \tag{8}$$

is the related *Lagrangian density* and

$$L_\mu = U^{-1}\partial_\mu U \tag{9}$$

are *left chiral currents*, where $F \cong 123$ MeV is pion decay constant and a is dimensionless constant. *The first term in equation (8) is the SU(2) chiral model Lagrangian density, the second term is Skyrme term for stabilizing solitonic solution. The last term descibes the mass term where M_π is pion (meson) mass.*

Euler-Lagrange equation of SU(2) Skyrme model is derived from *least action principle*

$$\delta S = 0. \tag{10}$$

If we take *variation of action* to equation (7), we obtain

$$\delta S = \int d^4x \ \delta\mathcal{L}. \tag{11}$$

Substitute equation (8) into (11), and use least action principle (10) [1]

$$\partial_\mu\left(L^\mu - \frac{1}{a^2F^2}\left[L_\nu, [L^\mu, L^\nu]\right]\right) + \frac{1}{2}M_\pi^2(U - U^{-1}) = 0. \tag{12}$$

In general case (nonstatic), energy of SU(2) Skyrme model is

$$E = \int d^3x \ T^{00}, \tag{13}$$

where T^{00} is *energy-momentum tensor*. Explicitly,

$$\begin{aligned}
E &= \int d^3x \ \text{Tr}\left[-\frac{F^2}{16}L_a L_a - \frac{1}{32a^2}[L_a, L_c][L_a, L_c]\right. \\
&\quad + \frac{F^2}{16}L_o L_o + \frac{1}{16a^2}[L_o, L_a][L_o, L_a] - \frac{F^2}{16}M_\pi^2(U^{-1} + U - 2I) - \frac{F^2}{8}L_o L_o \\
&\quad \left. - \frac{1}{8a^2}[L_a, L_o][L_a, L_o]\right] \\
&= E_{\text{static}} + E_{\text{rotation}}
\end{aligned} \tag{14}$$

where

$$E_{\text{static}} = -\int d^3x \ \text{Tr}\left[\frac{F^2}{16}L_a^2 + \frac{1}{32a^2}\left[L_a, L_c\right]^2 + \frac{F^2}{16}M_\pi^2(U^{-1} + U - 2I)\right] \tag{15}$$

and

$$E_{\text{rotation}} = -\int d^3x \ \text{Tr}\left[\frac{F^2}{16}L_0^2 + \frac{1}{16a^2}\left[L_0, L_a\right]^2\right]. \tag{16}$$

5

Solitonic properties of SU(2) Skyrme model for static energy in equation (15) is studied by scaling the spatial coordinates

$$x \to 2\widetilde{x}/aF$$

and express energy in $F/4a$, i.e. by taking

$$(F/4a) = (1/12\pi^2).$$

In this unit, equation (15) becomes

$$E_{\text{static}} = \frac{1}{12\pi^2} \int d^3x \left(-\frac{1}{2}\right) \text{Tr}\left[L_a^2 + \frac{1}{8}\left([L_a, L_c]^2 + m_\pi^2(U^{-1} + U - 2I)\right)\right] \tag{17}$$

where

$$m_\pi = 2M_\pi/aF. \tag{18}$$

Euler-Lagrange equation (12) in static case becomes

$$\partial_a \left(L_a - \frac{1}{4}\left[L_c, [L_a, L_c]\right]\right) - \frac{m_\pi^2}{2}(U - U^{-1}) = 0. \tag{19}$$

5 Scale Stability

Let us look at scale transformation below

$$x \to \lambda x. \tag{20}$$

We find that L_a currents, by scale transformation (20), transform into

$$L_a(x) \to U^{-1}(\lambda x)\frac{\partial U(\lambda x)}{\partial x^a} = \lambda L_a(\lambda x). \tag{21}$$

The effect of scale transformation to static energy (17), by ignoring pion mass term (because pion mass is small), is

$$E\left[\lambda\right]_{\text{static}} = \frac{1}{\lambda}E_\sigma + \lambda E_{\text{Sky}} \tag{22}$$

where E_σ is *static chiral energy term* and E_{Sky} is *Skyrme energy term*.

From equation (22), we obtain

$$\left.\frac{dE\left[\lambda\right]}{d\lambda}\right|_{\lambda=1} = \left(-\frac{1}{\lambda^2}E_\sigma + E_{\text{Sky}}\right)\bigg|_{\lambda=1} = -E_\sigma + E_{\text{Sky}}, \tag{23}$$

and

$$\left.\frac{d^2E\left[\lambda\right]}{d\lambda^2}\right|_{\lambda=1} = \frac{2}{\lambda^3}\left.E_\sigma\right|_{\lambda=1} = 2E_\sigma. \tag{24}$$

The requirement for *extremum condition* is

$$\frac{dE\left[\lambda\right]}{d\lambda} = 0. \tag{25}$$

We apply extremum condition, (25), to equation (23), we obtain

$$E_\sigma = E_{Sky}, \tag{26}$$

which it shows

$$E_\sigma \geq 0. \tag{27}$$

So that equation (24) fulfills condition

$$\frac{d^2 E\left[\lambda\right]}{d\lambda^2} > 0. \tag{28}$$

Equation (28) is *minimum stable condition* which implies that *static energy (22) is stable against scale perturbation.*

6 Topological Charge

Static energy of SU(2) Skyrme model can be expressed, by ignoring pion mass, as

$$E_{\text{static}} = \frac{1}{12\pi^2} \int d^3x \left(-\frac{1}{2}\right) \text{Tr}\left[\left(L_a \pm \frac{1}{4}\epsilon_{abc}\left[L_b, L_c\right]\right)^2\right] \mp \frac{1}{24\pi^2} \int d^3x\ \epsilon_{abc}\text{Tr}\left[L_a L_b L_c\right] \tag{29}$$

where ϵ_{abc} is Levi-Civita symbol, $\epsilon_{abc} = \delta_{123}^{abc}$. At energy lower bound

$$E_{\text{static}} \geq B, \tag{30}$$

where

$$B = -\frac{1}{24\pi^2} \int d^3x \epsilon_{abc}\text{Tr}\left(L_a L_b L_c\right). \tag{31}$$

B integral, (31), is independent of space-metric tensor, i.e. topological quantity which is known as *topological charge* of SU(2) Skyrme model.

7 Static Skyrme Equation in Spherical Coordinate

In spherical coordinate, (r, θ, ϕ), static Skyrme equation has the following form [1]

$$\begin{aligned}
0 = &\ \partial_r \left(L_r - \frac{1}{4}\left\{\frac{1}{r^2}[L_\theta, [L_r, L_\theta]] + \frac{1}{r^2 \sin^2\theta}[L_\phi, [L_r, L_\phi]]\right\}\right) \\
&+ \partial_\theta \left(\frac{1}{r^2}L_\theta - \frac{1}{4}\left\{\frac{1}{r^2}[L_r, [L_\theta, L_r]] + \frac{1}{r^4 \sin^2\theta}[L_\phi, [L_\theta, L_\phi]]\right\}\right) \\
&+ \partial_\phi \left(\frac{1}{r^2 \sin^2\theta}L_\phi - \frac{1}{4}\left\{\frac{1}{r^2 \sin^2\theta}[L_r, [L_\phi, L_r]] + \frac{1}{r^4 \sin^2\theta}[L_\theta, [L_\phi, L_\theta]]\right\}\right).
\end{aligned} \tag{32}$$

Its solution takes form

$$U(r) = \exp(iF_a(r, \theta, \phi)\sigma_a), \tag{33}$$

where

$$F_a(r, \theta, \phi) = g(r)n_a(\theta, \phi) \tag{34}$$

is Skyrme ansatz, $g(r)$ is profile function, n_a, $a = 1, 2, 3$ is component of unit vector, $\widehat{\boldsymbol{n}}$, in the internal space of SU(2), and σ_a is Pauli matrix.

8 Skyrmion Static Energy and Its Solution

As a result of scaling spatial coordinates, by ignoring pion mass, static energy of SU(2) Skyrme model can be stated as [1]

$$E_{\text{static}} = \frac{1}{12\pi^2}(4\pi)\int r^2 dr \left[\left(\frac{dg}{dr}\right)^2 + \frac{2}{r^2}\sin^2 g\left(1+\left(\frac{dg}{dr}\right)^2\right) + \frac{1}{r^4}\sin^4 g\right]. \tag{35}$$

From equation (35), by using least action principle

$$\delta_g E_{static} = 0, \tag{36}$$

we can derive Euler-Lagrange equation for profile function [1]

$$\frac{d^2 g}{dr^2}\left[1+\frac{2}{r^2}\sin^2 g\right] + \left(\frac{dg}{dr}\right)^2\left[\frac{1}{r^2}\sin 2g\right] + \left(\frac{dg}{dr}\right)\left(\frac{2}{r}\right) - \frac{1}{r^2}\sin 2g - \frac{1}{r^4}\sin^2 g\sin 2g = 0. \tag{37}$$

Equation (37) is second order of nonlinear differential equation.

The solution of equation (37) will be worked numerically with finite difference method. In order that static energy (35) has finite value at $r = 0$ and $r = \infty$, then profile function $g(r)$ must fulfills *boundary conditions*

$$g(0) = \pi, \quad g(\infty) = 0. \tag{38}$$

9 Quantized Rotational Energy

Skyrmion quantization is worked by involving time dependent of $U(\boldsymbol{r})$ as

$$U(\boldsymbol{r}) \to U(\boldsymbol{r},t) = A(t)U(\boldsymbol{r})A(t)^\dagger, \tag{39}$$

where

$$A(t) \in SU(2)_{\text{internal}},$$
$$AA^\dagger = A^\dagger A = I \tag{40}$$

A is time dependent unitary matrix.

By using equation (39), rotational energy can be stated as [1], [12]

$$\begin{aligned}
E_{\text{rotation}} &= -\left(\frac{\pi F^2}{6}\int_0^\infty dr\, r^2\sin^2 g + \frac{2\pi}{3a^2}\int_0^\infty dr\, r^2\sin^2 g\left[\left(\frac{dg}{dr}\right)^2 + \frac{1}{r^2}\sin^2 g\right]\right)\text{Tr}\left(R^{-1}\frac{\partial R}{\partial t}\right)^2 \\
&= \frac{1}{2}\, I\,\text{Tr}\,\boldsymbol{\Omega}^2
\end{aligned} \tag{41}$$

where

$$\text{Tr}\,\boldsymbol{\Omega}^2 = -\text{Tr}\left(R^{-1}\frac{\partial R}{\partial t}\right)^2 \tag{42}$$

with $\boldsymbol{\Omega}$ is angular velocity matrix of Skyrmion, and

$$I = 2\left(\frac{\pi F^2}{6}\int_0^\infty dr\, r^2\sin^2 g + \frac{2\pi}{3a^2}\int_0^\infty dr\, r^2\sin^2 g\left[\left(\frac{dg}{dr}\right)^2 + \frac{1}{r^2}\sin^2 g\right]\right) \tag{43}$$

is Skyrmion moment of inertia.

10 Finite Difference Method

Assume that $g(x)$ is a continuous function of one variable. The value of this function is given only for discrete value, and equidistant x [20]:

$$g_k \equiv g(x_k), \tag{44}$$

where $x_k \equiv x_0 + k\Delta x$. Let us define quantity

$$\Delta g_k \equiv g_{k+1} - g_k \tag{45}$$

which is called *forward difference* at point x_k. Apply (45) for higher order of forward difference, we get

$$\Delta^2 g_k \equiv \Delta g_{k+1} - \Delta g_k = g_{k+2} - 2g_{k+1} + g_k, \tag{46}$$

$$\Delta^3 g_k \equiv \Delta^2 g_{k+1} - \Delta^2 g_k = g_{k+3} - 3g_{k+2} + 3g_{k+1} - g_k \tag{47}$$

and so on. General form of forward difference is:

$$\Delta^r g_k \equiv \Sigma_{i=0}^{r} (-1)^i \binom{r}{i} g_{k+r-i}. \tag{48}$$

Backward difference is defined as

$$\nabla g_k \equiv g_k - g_{k-1} \tag{49}$$

and higher order of backward difference is

$$\nabla^2 g_k \equiv \nabla g_k - \nabla g_{k-1} = g_k - 2g_{k-1} + g_{k-2} \tag{50}$$

so on. General form of backward difference is

$$\nabla^r g_k \equiv \Sigma_{i=0}^{r} (-1)^i \binom{r}{i} g_{k-r+i}. \tag{51}$$

Following, it is defined *central difference* which is symmetric with x_k:

$$\delta g_k \equiv g_{k+1/2} - g_{k-1/2} \tag{52}$$

and higher order of central difference is

$$\delta^2 g_k = g_{k+1} - 2g_k + g_{k-1} \tag{53}$$

so on. General form of central difference is

$$\delta^r g_k \equiv \Sigma_{i=0}^{r} (-1)^i \binom{r}{i} g_{k-r/2+i}. \tag{54}$$

If central difference is observed, $k+1/2$ index is out of the value which is served by discrete value. In order to obtain the integer value, mean value operator, μ, is used to operate with $g(x)$ function as *central mean*

$$\mu g = \frac{1}{2} \left[g_{k+1/2} + g_{k-1/2} \right] \tag{55}$$

and higher order of central mean is

$$\mu^2 g_k \equiv \frac{1}{2} \left[\mu g_{k+1/2} + \mu g_{k-1/2} \right] \tag{56}$$

so on. *Central mean of central difference* is defined as

$$\mu \delta g_k \equiv \frac{1}{2} \left[\delta g_{k+1/2} + \delta g_{k-1/2} \right]$$
$$= \frac{1}{2} \left[g_{k+1} - g_{k-1} \right]. \tag{57}$$

11 Numerical Solution of Skyrme Equation

Skyrme equation is stated in finite difference form, i.e. by substituting equation (49), (53), into Skyrme equation (37) then we obtain

$$\frac{\delta^2 g_k}{\Delta r^2} \left[1 + \frac{2}{r^2} \sin^2 g_k \right] + \left(\frac{\nabla g_k}{\Delta r} \right)^2 \left[\frac{1}{r^2} \sin\ 2g_k \right] + \left(\frac{\nabla g_k}{\Delta r} \right) \left(\frac{2}{r} \right) - \frac{1}{r^2} \sin\ 2g_k - \frac{1}{r^4} \sin^2 g_k\ \sin\ 2g_k = 0. \tag{58}$$

By reposisition of equation (58), we get

$$\delta^2 g_k = \frac{- \left(\frac{\nabla g_k}{\Delta r} \right)^2 \left[\frac{1}{r^2} \sin\ 2g_k \right] - \left(\frac{\nabla g_k}{\Delta r} \right) \left(\frac{2}{r} \right) + \frac{1}{r^2} \sin\ 2g_k + \frac{1}{r^4} \sin^2 g_k \sin\ 2g_k}{\left[1 + \frac{2}{r^2} \sin^2 g_k \right]} \Delta r^2$$
$$= \frac{-(\nabla g_k)^2 \sin\ 2g_k - (\nabla g_k)2(\Delta r)r + \sin\ 2g_k \Delta r^2 + \frac{1}{r^2} \sin^2 g_k \sin\ 2g_k \Delta r^2}{r^2 + 2 \sin^2 g_k} \tag{59}$$

$$g_{k+1} = \frac{-(\nabla g_k)^2 \sin\ 2g_k - (\nabla g_k)2(\Delta r)r + \sin\ 2g_k \Delta r^2 + \frac{1}{r^2} \sin^2 g_k \sin\ 2g_k \Delta r^2}{r^2 + 2 \sin^2 g_k}$$
$$- g_{k-1} + 2g_k. \tag{60}$$

The problems are initial values which are needed to be defined, i.e.

$$g(r = 0), \quad g(r = dr). \tag{61}$$

Boundary conditions, which are given by Skyrme, are

$$g(r = 0) = \pi, \quad g(r \to \infty) = 0. \tag{62}$$

Apply the boundary conditions (62) for initial values (61). It is suitable for $g(r = 0)$, and $g(r = dr)$ must be chosen, so that it is suitable for $g(r \to \infty) = 0$. This method is known as *shooting method* [13]. The ploting result, as shown in Figure 1: Profile function $\phi(r)$, can be compared with trial and error method, i.e.

$$g(x) = 4 \arctan\ e^{-r}. \tag{63}$$

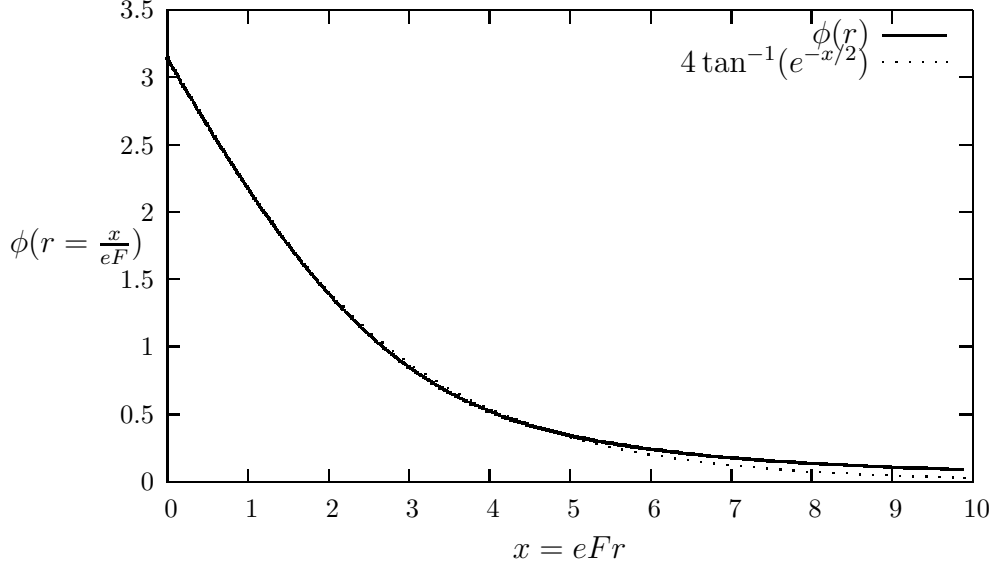In order to calculate static energy (35), the following integral is used

Figure 1: Profile function $\phi(r)$

$$E(g) = \frac{2\pi f_\pi}{e} \int_0^\infty dr \left[\left(r^2 g'^2 + 2\sin^2 g \right) + \sin^2 g \left(2g'^2 + \frac{\sin^2 g}{r^2} \right) \right] \qquad (64)$$

where $g' = dg/dr$. Equation (64), is solved numerically with *trapezoid method* and it gives:

$$23.2154 \ \pi f_\pi / e.$$

Let us calculate numerically E_{static} (35), using numerical value of $g(r)$

$$E_{\text{static}} = \frac{1}{12\pi^2} (4\pi) \left(\frac{23.2}{2} \right). \qquad (65)$$

In the form of F and a, using definition $(1/12\pi^2) = (F/4a)$, we obtain

$$E_{\text{static}} = \frac{\pi F}{a} \left(\frac{23.2}{2} \right). \qquad (66)$$

The numerical value of $g(r)$ can be used to calculate

(1) nucleon (m_N) and delta static masses (m_Δ);

(2) Skyrmion moment of inertia, I.

We get benefit from numerical value of profile function, $g(r)$, to calculate Skyrmion moment of inertia. It gives

$$I = \frac{1}{4} \left(\frac{447}{Fa^3} \right). \qquad (67)$$

In quantum mechanics, angular momentum $\boldsymbol{J}$ is quantized as

$$\boldsymbol{J}^2 = j(j+1)\hbar^2 \qquad (68)$$

11

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$$\hbar = \frac{h}{2\pi},$$

h is Planck constant. Here, we use using natural unit, $\hbar = 1 = c$.

Rotational energy is

$$E_{\text{rotation}} = \frac{\boldsymbol{J}^2}{2I} = \frac{j(j+1)}{2I}. \tag{69}$$

12 Hadron Mass

Skyrmion is solution of Skyrme equation which has finite energy. Based on Einstein energy-mass formula, Skyrmion mass is

$$m = \frac{E_{static} + E_{rotation}}{c^2} = M_{static} + M_{rotation}. \tag{70}$$

From equation (69) and (70), we obtain

$$m = M_{static} + \frac{j(j+1)}{2I}. \tag{71}$$

Wess-Zumino quantization condition [3], requires:

$$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

It means that Skyrmion is fermion.

In case of nucleon with spin $\frac{1}{2}$, $j = \frac{1}{2}$, equation (71) gives

$$m_N = M_{static} + \frac{\frac{1}{2}(\frac{1}{2}+1)}{2I} \tag{72}$$

By substituting the values of $F = 123$ MeV and $a = 4.95$ into (66) and (67), give

$$M_{static} \cong 905.08 \text{ MeV} \tag{73}$$

and

$$I \cong 7.49 \times 10^{-3}. \tag{74}$$

Substitute (73) and (74) into (72) gives the result

$$m_N \cong 955.15 \text{ MeV}. \tag{75}$$

In case of delta particle with spin $\frac{3}{2}$, $j = \frac{3}{2}$, equation (71) gives

$$m_\Delta = M_{static} + \frac{\frac{3}{2}(\frac{3}{2}+1)}{2I}. \tag{76}$$

By substituting the values of M_{static} and I above, it gives

$$m_\Delta \cong 1155.41 \text{ MeV}. \tag{77}$$

13 Discussions

The calculation of nucleon energy-mass gives the value about 955.15 MeV. The experimental value of nucleon energy-mass is about 939 MeV. The different value of nucleon energy-mass based on calculation and experiment is about 16.15 MeV. The calculation of delta energy-mass gives the value about 1155.41 MeV. The experimental value of delta energy-mass is about 1232 MeV [3]. The different value of delta energy-mass based on calculation and experiment is about 76.59 MeV.

The different values among calculation and experiment are becaused of SU(2) Skyrme model is very simple model, i.e. it consists of two flavours, (u, d) or (u, s) or (d, s). It is necessary keeping in mind related with these different values [15]:

(1) Meson field Lagrangian only includes pseudoscalar field. Other low mass meson (vector) should be included.

(2) In nature, there are three flavour families than two "light" flavour families which must be included in more realistic formula.

(3) The effects of chiral and flavour symmetry breaking aren't calculated yet.

(4) N_C (number of colour) correction of nucleon mass isn't included yet.

14 Acknowledgment

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15   Appendix: Flow Chart of Skyrme Profile Function Calculation and Its Integration

```
clear;
clf;
okmainloop=0;
N=1;
dr=.001;
$%$dr=.01;
```

```
rf=50;
%rf=8;
rf=50;
rf=80;

nnl=length([0 dr:dr:rf rf+dr]);
g(nnl)=0;
n0=1;
n=2;
%%%%%%%%
v=1+.0041
v0=v;
g(n0)=pi;%+0.01;
g(n)=g(n0)-v*dr;%0.02;

if okmainloop
% Looping utama program ------------
for r=[dr:dr:rf]
    n=n0+1;
    n1=n+1;

    gg=g(n);
    bg=g(n)-g(n0);
    bg2=bg*bg;
    sin2g=sin(gg)*sin(gg);
    sing2=sin(2*gg);
    r2=r*r;
    dr2=dr*dr;

    % x=eFr  (catto)
    g(n1)=(-2*r*bg*dr-sing2*(4*bg2-4*sin2g*dr2/r2-dr2))/(r2+8*sin2g)+2*g(n)-g(n0);

    n0=n0+1;
    %if (g(n1)<=0) break; end;
end
% akhir looping utama----------------
save simpan.mat
else
load simpan.mat
end

rr=[0 dr:dr:rf rf+dr];
nn=n1;
gg=g(1:n1);
```

```
%subplot(2,1,1);
subplot(1,1,1);
plot(rr(1:nn-1),gg(1:nn-1),'b-');

rr=[0 dr:dr:rf rf+dr];
%rre=pi*exp(-rr);
rre=4*atan(exp(-rr));

hold on;
plot(rr(1:nn-1),rre(1:nn-1),'r-');
grid on

%%%%%% negatif change to zero
%g=g.*(g>0);

% Statical calculation
IIEUc=0;
IIaUc=0;
IIbUc=0;
IICUc=0;
IIrUc=0;


jumlah=0;
dgdr2=v*v; % from initial value
gg=g(1);
sin2g=sin(gg)*sin(gg);


for n=[1:nnl-1]
   dg=g(n+1)-g(n);
   %dg1=g(n+2)-g(n);
    if n==1
        dgdr2=(dg*dg)/(dr*dr);
    else
        bg=g(n)-g(n-1);
        Dg=(g(n+1)-g(n-1))*.5;
        dgdr2=(dg*dg)/(dr*dr);
        %dgdr2=(dg*bg)/(dr*dr);
        %dgdr2=(Dg*Dg)/(dr*dr);

    end
    gg=g(n);
    r=n*dr;
```

```matlab
    sin2g=sin(gg)*sin(gg);

    jumlah=jumlah+dr*gg;
    r2=r*r;
    IIEUc=IIEUc+ dr*( r2*dgdr2 + sin2g*(2+8*dgdr2+4*sin2g/(r*r)) );  % catto
    IIaUc=IIaUc+ dr*r2*sin2g*( 1 +  4*(dgdr2 + sin2g/(r*r))  );  % catto
    IIbUc=IIbUc+ dr*r2*(1-cos(gg))*(1 + dgdr2 + 2*sin2g/(r*r));
    IICUc=IICUc+ dr*r2*(1-cos(gg));
    IIrUc=IIrUc+ dr*r2*sin2g*(bg/dr);
    if (g(n+1)<0) break; end
end

IEUc=IIEUc*pi*4/8;
IaUc=IIaUc*pi*2/3;
IbUc=IIbUc*pi*1/2;
ICUc=IICUc*pi*1/2;
IrUc=IIrUc*2/pi;

save II.mat II* dr v0
```